Pattern synthesis of sparse linear array by off-grid Bayesian compressive sampling

Jincheng Lin✉, Xiaochuan Ma, Shefeng Yan and Li Jiang

An off-grid (OG) pattern synthesis algorithm for sparse non-uniform linear arrays is presented. It is based on Bayesian compressive sampling (BCS), and the design of maximally sparse linear arrays for the given reference patterns can be obtained. The proposed algorithm novelly introduces the OG model into the pattern synthesis problem, and it makes the synthesis more accurate than the conventional BCS algorithm. Moreover, the proposed algorithm has the advantage of high computational efficiency, since the BCS-based algorithms can be realised by the fast relevance vector machine. Numerical experiments show that the proposed algorithm has improved accuracy in terms of normalised mean square error.

Introduction: Pattern synthesis for antenna arrays with a minimum number of elements is a problem of great importance in many applications (e.g. radar, remote sensing and wireless communication). Among pattern synthesis techniques, non-uniformly spaced arrays [1, 2] have potential advantages with respect to uniform arrangements such as decreased mainlobe width and sidelobe suppression. Recently, an effective pattern synthesis method based on Bayesian compressive sampling (BCS) with sparse non-uniform linear arrays (SNLAs) has been proposed [1, 3]. This method has the significant advantages of flexibility and computational efficiency. Actually, the SNLA in [1] can be considered as a uniform linear array (ULA) with 'missing' array elements. However, the BCS-based method in [1, 3] has a limitation that the positions of sparse array elements must be selected from a given virtual ULA. Uniformly spaced elements of the ULA separate the array aperture on a fixed uniform grid. Thus, this approach may induce an unideal approximation to the optimal element position. The maximum deviation can reach $\Delta d/2$, where Δd is the element spacing of the virtual ULA. As a matter of fact, the deviation caused by the fixed grid has been noticed in many problems such as direction-of-arrival estimation, spectral estimation and signal reconstruction.

This Letter introduces a new off-grid (OG) synthesis method which can estimate the sparse array element positions more accurately and synthesise the pattern with less error than the existing BCS-based method. Precisely, the new pattern synthesis method can guarantee the sparsity of the SNLA due to the property of BCS and provide a more accurate fit of the given pattern (e.g. Chebyshev, Taylor-Kaiser) than the conventional BCS synthesis.

OG model for pattern synthesis: Given a reference pattern $P_{\mathrm{REF}}(\theta)$, $\theta \in [0°,\ 180°]$, the target of pattern synthesis is to find a set of array element positions $\{d_m\}_{m=1}^{M}$ and their corresponding weights in order to satisfy

$$P_{\mathrm{REF}}(u) = \sum_{m=1}^{M} w_m \exp(\mathrm{j}kd_m u), \quad u \in [-1,\ 1] \tag{1}$$

where $u = \cos\theta$, $k = 2\pi/\lambda$ is the wavenumber, d_m denotes the position of the mth element and its corresponding weight is w_m. We can evaluate d_m and w_m from a set of L samples of the reference pattern; precisely, $y_l = P_{\mathrm{REF}}(u_l)$, $l \in \{1,\ \ldots,\ L\}$. In [1], the reconstruction of $\{d_m,\ w_m\}_{m=1}^{M}$ from the pattern sample set $\{y_l\}_{l=1}^{L}$ is considered as the linear regression from an overcomplete set of $\{\tilde{d}_n,\ w_n\}_{n=1}^{N}$, where $N \gg M$. Precisely

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{w} + \varepsilon \tag{2}$$

where $\boldsymbol{y} = [y_1,\ \ldots,\ y_L]^{\mathrm{T}}$, $\boldsymbol{A} = \left[\boldsymbol{a}(\tilde{d}_1),\ \ldots,\ \boldsymbol{a}(\tilde{d}_N)\right]$, $\boldsymbol{w} = [w_1,\ \ldots,\ w_N]^{\mathrm{T}}$ and $\varepsilon = [\varepsilon_1,\ \ldots,\ \varepsilon_L]^{\mathrm{T}}$. The (l, n)th entry of $\boldsymbol{A}$ is $a_l(\tilde{d}_n) = \exp(\mathrm{j}k\tilde{d}_n u_l)$ and $\tilde{d}_n$ is the nth element position of the user-chosen overcomplete array element set. Without loss of generality, $\tilde{d}_n$ can be selected from a uniform grid and the spacing is denoted as $\Delta d = \tilde{d}_n - \tilde{d}_{n-1}$. Though the existing BCS-based approach has shown improvement in pattern synthesis, e.g. success in reducing the number of array elements, there are still more or less deviations when the optimal array element positions are not on the sampling position grid.

Suppose $d_m \neq \{\tilde{d}_1,\ \ldots,\ \tilde{d}_N\}$ for some $m \in \{1,\ \ldots,\ M\}$, there is the deviation of $\boldsymbol{a}(d_m) - \boldsymbol{a}(\tilde{d}_{n_m})$ even if $\tilde{d}_{n_m}$ is the nearest grid point to d_m. Inspired by the OG technique in [4], we approximate $\boldsymbol{a}(d_m)$ using linearisation

$$\boldsymbol{a}(d_m) \approx \boldsymbol{a}(\tilde{d}_{n_m}) + \boldsymbol{b}(\tilde{d}_{n_m})(d_m - \tilde{d}_{n_m}) \tag{3}$$

where $\boldsymbol{b}(\tilde{d}_{n_m}) = \boldsymbol{a}'(\tilde{d}_{n_m})$, namely $b_l(\tilde{d}_{n_m}) = \mathrm{j}ku_l \exp(\mathrm{j}k\tilde{d}_{n_m}u_l)$. Actually the OG approximation in (3) is the Taylor first-order approximation which is more accurate than the zero-order one. By denoting $\boldsymbol{B} = \left[\boldsymbol{b}(\tilde{d}_1),\ \ldots,\ \boldsymbol{b}(\tilde{d}_N)\right]$ and $\boldsymbol{\Phi} = \boldsymbol{A} + \boldsymbol{B}\,\mathrm{diag}(\boldsymbol{\beta})$, we can write the observation model of (2) into $\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{w} + \varepsilon_{\Phi}$, which is the OG model used in this Letter.

Pattern synthesis via OG-BCS: To design a SNLA which is maximally sparse, we convert the pattern synthesis problem of (2) into the compressive sensing (CS) problem as

$$\min ||\boldsymbol{w}||_1 \quad \text{s.t.} ||\boldsymbol{y} - \boldsymbol{A}\boldsymbol{w}||_2^2 \leq \boldsymbol{\epsilon} \tag{4}$$

According to the BCS theory [5, 6], the sparse Bayesian learning algorithm can provide a tighter approximation to the ℓ_0-norm sparsity than the ℓ_1-norm in (4). The objective is to find $\widehat{\boldsymbol{w}}$ in order to maximise the hierarchical posterior

$$\widehat{\boldsymbol{w}} = \arg\max_{\boldsymbol{w}} p_{\mathrm{BCS}}(\boldsymbol{w},\ \boldsymbol{\alpha},\ \alpha_0|\boldsymbol{y}) \tag{5}$$

where $\boldsymbol{\alpha}$ is the hyperparameter of the hierarchical prior model, α_0^{-1} denotes the variance of ε and $p_{\mathrm{BCS}}(\cdot)$ represents the posteriori probability of the hierarchical Bayesian model. The detail of the BCS approach is shown in Fig. 1.

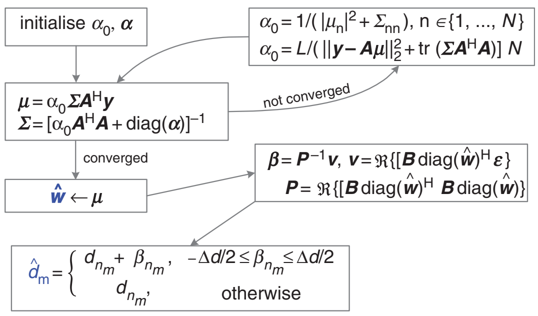

Fig. 1 *Off-grid pattern synthesis method based on BCS*

After obtaining $\widehat{\boldsymbol{w}}$, the adjustment parameters $\boldsymbol{\beta}$ of the OG model need to be estimated. Actually an appropriate $\boldsymbol{\beta}$ should minimise the pattern synthesis error ε_{Φ}, namely

$$\min_{\boldsymbol{\beta}} ||\boldsymbol{y} - (\boldsymbol{A} + \boldsymbol{B}\,\mathrm{diag}(\boldsymbol{\beta}))\widehat{\boldsymbol{w}}||_2^2 \tag{6}$$

Because $\boldsymbol{B}\mathrm{diag}(\boldsymbol{\beta})\widehat{\boldsymbol{w}} = \boldsymbol{B}\mathrm{diag}(\widehat{\boldsymbol{w}})\boldsymbol{\beta} = \boldsymbol{B}_w$, we rewrite (6) as follows:

$$\arg\min_{\boldsymbol{\beta}} \{\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{B}_w^{\mathrm{H}}\boldsymbol{B}_w\boldsymbol{\beta} - \varepsilon^{\mathrm{H}}\boldsymbol{B}_w\boldsymbol{\beta} - \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{B}_w^{\mathrm{H}}\varepsilon + \varepsilon^{\mathrm{H}}\varepsilon\} \tag{7}$$

Because $\boldsymbol{\beta} \in \mathbb{R}^N$, $\partial||\varepsilon - \boldsymbol{B}_w\beta||_2^2/\partial\boldsymbol{\beta} = 0$ results in $\boldsymbol{\beta} = \boldsymbol{P}^{-1}\boldsymbol{v}$, where $\boldsymbol{P} = \Re\{\boldsymbol{B}_w^{\mathrm{H}}\boldsymbol{B}_w\}$, $\boldsymbol{v} = \Re\{\boldsymbol{B}_w^{\mathrm{H}}\varepsilon\}$. To guarantee $\tilde{d}_{n_m}$ the nearest grid point to d_m, we constraint $\beta_{n_m} \in [-\Delta d/2,\ \Delta d/2]$. The OG-BCS algorithm for pattern synthesis is detailed in Fig. 1.

Simulation results: In this Section, two kinds of patterns, i.e. Dolph-Chebyshev and Taylor-Kaiser, are considered as the reference patterns. In Figs. 2*a*,*b* and *c*,*d* the apertures and sidelobe levels (SLLs) of the Dolph-Chebyshev and the (Taylor-Kaiser) patterns are {9.5 λ, −30 dB} and {19.5 λ, −30 dB}, respectively. As we know, the Dolph-Chebyshev reference patterns in Figs. 2*a* and *b* can be synthesised through the uniform arrays with 20 and 40 λ/2-spaced elements, respectively, and the parameters of the Taylor-Kaiser reference patterns in Figs. 2*c* and *d* are the same. In all of our simulations, the OG-BCS algorithm for pattern synthesis is compared with the conventional BCS algorithm and the reference patterns. The BCS and OG-BCS synthesis are carried out by sampling $P_{\mathrm{REF}}(u)$ at L points, i.e. $u_l = (2l - L - 1)/L$, $l \in \{1,\ \ldots,\ L\}$, and the overcomplete array element positions $d_n = (2n - N - 2)$aperture/$2N$, $n \in \{1,\ \ldots,\ N\}$. From Figs. 2*a*–*d*, it can be observed that the fitting degree of OG-BCS to the reference pattern (Dolph-Chebyshev or Taylor-Kaiser) has been improved compared with the BCS algorithm. Moreover, the OG-BCS algorithm has the same excellent sparsity as the conventional BCS. The numbers of active elements of the OG-BCS solution with respect to two array apertures 9.5 and 19.5λ are 14 and 26, respectively. In contrast, the regular Dolph-Chebyshev or Taylor-Kaiser patterns for 9.5 and 19.5λ array apertures need to be synthesised by 20 and 40 uniformly spaced array elements, respectively. As shown in Figs. 2*a* and *c*, the proposed synthesis algorithm can save 30% array elements, and in Figs. 2*b* and *d* it is 35%.

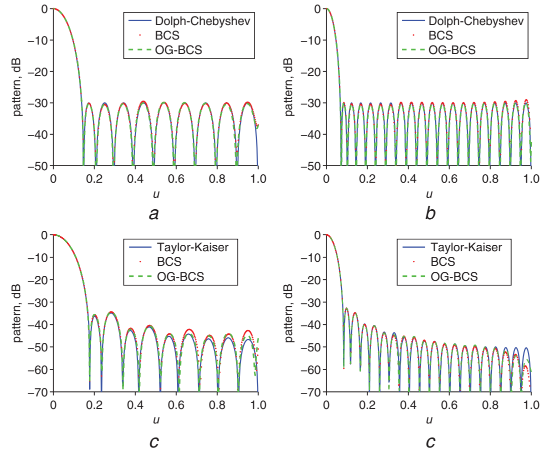

Fig. 2 *Reference patterns*

a, *b* Dolph-Chebyshev
c, d Taylor-Kaiser
Array element numbers of SNLAs designed by OG-BCS and BCS are 14 [Figs. 2*a*,*c*] and 26 [Figs. 2*b*,*d*]. SLL is set as −30 dB

To quantitatively evaluate the accuracies of the different methods, the following normalised mean square error (NMSE) is used as the metric

$$\mathrm{NMSE} = \frac{\int_{-1}^{1} |\widehat{P}(u) - P_{\mathrm{REF}}(u)|^2 \,\mathrm{d}u}{\int_{-1}^{1} |P_{\mathrm{REF}}(u)|^2 \,\mathrm{d}u} \quad (8)$$

where $\widehat{P}(\cdot)$ can be $P_{\mathrm{BCS}}(\cdot)$ or $P_{\mathrm{OG-BCS}}(\cdot)$. Figs. 3*a*–*d* indicate the NMSEs of OG-BCS and BCS with respect to the reference patterns (i.e. Dolph-Chebyshev and Taylor-Kaiser) under different SLLs. Figs. 3*a* and *b* (Figs. 3*c* and *d*) demonstrate the NMSEs for the synthesis of Dolph-Chebyshev (Taylor-Kaiser) patterns with 20 and 40 $\lambda/2$-spaced array elements. It can be observed that the OG-BCS algorithm method usually has a lower RMSE than the BCS algorithm under different SLLs (i.e. −20 to −40 dB) and array apertures (i.e. 9.5 and 19.5λ).

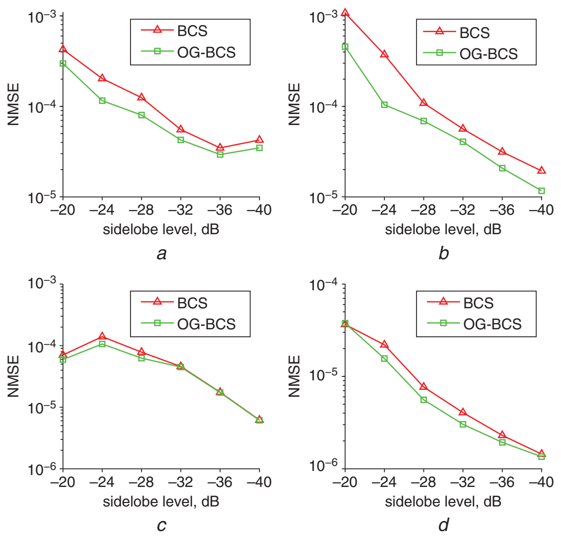

Fig. 3 *OG-BCS and BCS assessments (NMSE) against variant SLLs (−20 to −40 dB)*

Reference patterns are:
a, *b* Dolph-Chebyshev
c, d Taylor-Kaiser
Numbers of SNLA elements are 14 [Figs. 3*a*,*c*] and 26 [Figs. 3*b*,*d*]

The reason why the OG-BCS algorithm outperforms the conventional BCS is that the adjustment parameters $\boldsymbol{\beta}$ of the OG model for pattern synthesis can provide a closer approximation to the optimal array element positions than the non-OG model. Moreover, the OG-BCS algorithm has the advantage of high-efficiency computation, because the BCS-based algorithms can be realised by a fast relevance vector machine (RVM) [6, 7].

Conclusion: In this Letter, a high-accuracy and maximally sparse pattern synthesis method is proposed. Given a set of L samples of the reference pattern, the proposed method converts the pattern synthesis problem into the sparse signal restoration from an overcomplete set of $\{d_n, w_n\}_{n=1}^{N}$ ($N \gg L > M$). Owing to the sparsity of the BCS-based algorithms, the reference pattern can be synthesised by a much smaller set of sparse non-uniform array elements than the uniformly spaced linear array. Reduction of the array element number will usually make the array system more efficient. Furthermore, the proposed method utilises the OG model to estimate array element positions d_m, $m = 1, \ldots, M$ more accurately than the conventional model which uses a fixed uniform position grid. Another advantage of the proposed algorithm is that the BCS-based algorithms can be more efficiently achieved by a fast RVM than conventional CS algorithms. To summarise, the proposed OG BCS method for pattern synthesis has the advantages of sparsity, accuracy and high efficiency compared with existing state-of-the-art pattern synthesis methods.

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One or more of the Figures in this Letter are available in colour online.

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