THE ALGORITHM OF DISPERSION FUNCTION OF RAYLEIGH WAVES' IN POROUS LAYERED HALF-SPACE

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The dispersion functions of Rayleigh waves are theoretically deduced in layered half-space, which consist of elastic and porous layers. Different with the Rayleigh waves of elastic half-space, the dispersion function changes when the porous layer exists, and relates with the position of porous layers. The dispersion functions are given in cases of the porous layers locating at different positions. Furthermore, by the resolution of the original transfer matrix, an optimized square matrix of 15 orders is deduced in order to eliminate the cubic terms of *e*, this greatly improves the computable frequency range of Rayleigh waves' dispersion.

Keywords: Layered half-space; Porous layers; Rayleigh waves; Dispersion

1. INTRODUCTION

Porous formation is one of the most common formation in geophysical and engineering geophysical exploration, and one of the most important objects of oil and gas resources exploration. Therefore, getting accurate the parameters of porous media is a significant work of oil and gas resources exploration. The exploration by Rayleigh waves is one of the most effect engineering explorations, though, which founded on elastic layered half-space. It will make mistakes when we apply Rayleigh waves' exploration on a porous layered half-space. To solve this problem, we theoretically study on the excitation and propagation mechanism of Rayleigh waves in a porous layered half-space.

The transfer matrix method is always used in theoretical research of acoustic propagation in layered media. It is a simple and effective means to calculate the dispersion curves. However, this method has a limitationthat it may cause the significant digit losing in high frequency range. In the porous layer, the problem is even harder due to the slow longitude wave and the dissipation effect. Abo-zena[1] and Menke[2] proposed a methodology to solve the problem, Zhang[3] improved the methodology. However, these former researches are all based on elastic layered half-space, there is no effective way to solve the problem in porous media.

In this paper, we focus to theoretically derive the detail form of Rayleigh waves' dispersion function in layered half-space with a porous layer is located in different positions. Furthermore, we propose an optimized algorithm to improve the problem of transfer matrix method that it may lose significant digit in high frequency range. By this algorithm, both the accuracy and frequency range of the dispersion computation are all improved.

2. BOUNDARY CONDITIONS AND DISPERSION FUNCTION

The layered half-space is assumed as a model in any combination of elastic and porous layers, Fig.1 shows the configuration.

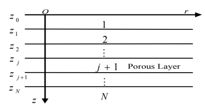

Figure 1. The configuration of layered half-space with porous layers

We introduce the (*b,p,c*) coordination system, for convenience. Assume $\mathbf{S}$ as the vector of displacement and stress spectrum in elastic media, its formation is

$$\mathbf{S} = (u_B / k, \tau_P / \omega^2, u_P / k, \tau_B / \omega^2)^T \tag{1}$$

where u_P, u_B are the components of displacement spectrum, τ_P, τ_B are the components of stress spectrum in the normal direction.Hence, assume the j^{th} layer is elastic, then the transfer relationship of stress and displacement spectrum between the top and bottom surface of the j^{th} layer is

$$\mathbf{S}_j(z_j) = M_j \lambda_j M_j^{-1} \mathbf{S}_j(z_{j-1}), \tag{2}$$

In a porous medium, the formation of S is

$$\mathbf{S}^d = (u_B^d / k, -P_f^d / \omega^2, \tau_P^d / \omega^2, u_P^d / k, w_P^d / k, \tau_B^d / \omega^2)^T, \tag{3}$$

where P_f^d is the pressure of fluid phase, u_P^d, u_B^d are the

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displacements of solid phase, w_P^d is the relative flow displacement, τ_P^d, τ_B^d are total stress spectrum of the sections of porous media. Therefore, the recursive relation can be obtained as following when the j^{th} layer is porous,

$$\mathbf{S}_j^d(z_j) = M_j^d \lambda_j^d (M_j^d)^{-1} \mathbf{S}_j^d(z_{j-1}), \tag{4}$$

For convenience, Eq. (2) and (4) are formatting as

$$\begin{cases} \mathbf{S}_j(z_j) = P_j \mathbf{S}_j(z_{j-1}) \\ \mathbf{S}_j^d(z_j) = H_j \mathbf{S}_j^d(z_{j-1}) \end{cases}, \tag{5}$$

The transfer matrix P_j is a 4 orders square matrix for the elastic layers, while H_j is a 6 orders square matrix for the porous layers. These matrices are with different orders, hence, we cannot directly compute the transfer function when porous and elastic layers exist in the layered half-space simultaneously. Furthermore, the boundary condition of the free surface and acoustic propagation are different, so we should deduce the new dispersion functions for the different relative positions of porous and elastic layers.

In (b, p, c) coordinate system, there is no coupled relation between $P-SV$ and SH waves, they can be processed separately by same way, so we discuss $P-SV$ wave only. The vectors consisted by potential functions in elastic and porous media are shown in follow,

$$\begin{cases} \boldsymbol{\varphi}(z) = (Ae^{iaz}, Be^{-iaz}, Ce^{ibz}, De^{-ibz})^T = (\varphi^+, \varphi^-, \psi^+, \psi^-)^T \\ \boldsymbol{\varphi}^d(z) = (A_{m1}e^{a_1 z}, B_{m1}e^{-a_1 z}, A_{m2}e^{a_2 z}, B_{m2}e^{-a_2 z}, C_m e^{bz}, D_m e^{-bz})^T \\ \quad = (\varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^-, \psi^+, \psi^-)^T \end{cases}, \tag{6}$$

where φ and ψ represent the potential function of P wave and SV wave, respectively. The superscript "$+$" and "$-$" represent the wave propagate along and against the z axis direction, φ_1 and φ_2 represent the first and second longitude waves' potential.

When the porous layer locates in the j+1$^{\text{th}}$ layer, the boundary conditionsof z_j are as follows:

$$\begin{cases} u_B^j(z_j) = u_B^{j+1}(z_j) \\ u_P^j(z_j) = u_P^{j+1}(z_j) \\ \tau_P^j(z_j) = \tau_P^{j+1}(z_j) \\ \tau_B^j(z_j) = \tau_B^{j+1}(z_j) \\ 0 = w_P^{j+1}(z_j) \end{cases}. \tag{7}$$

2.1 Porous layer is at the bottom

In this case, the porous layer is at the j+1th layer, and all the upper layers are elastic, so

$$\boldsymbol{\varphi}_d(z_{j+1}) = M_d^{-1} \mathbf{S}_d(z_j). \tag{8}$$

Based on the boundary conditions (the first four equations of Eq.(7)), we can obtain

$$\begin{bmatrix} u_B^{j+1}(z_j)/k \\ \tau_P^{j+1}(z_j)/\omega^2 \\ u_P^{j+1}(z_j)/k \\ \tau_B^{j+1}(z_j)/\omega^2 \end{bmatrix}_d = P(z_j, z_{j-1}) \cdots P(z_1, z_0)\mathbf{S}(z_0) = P(z_j, z_0)\mathbf{S}(z_0) \tag{9}$$

Build a matrix for $G = D \cdot P(z_j, z_0)$, the formation is

$$D = \begin{bmatrix} m'_{11} & \cdots & m'_{16} \\ \vdots & \ddots & \vdots \\ m'_{61} & \cdots & m'_{66} \end{bmatrix}, \tag{10}$$

where $m'_{ij} = \lfloor M_d{}' \rfloor_{ij}$.

Substitute Eq. (9) and the fifth equation of Eq. (7) into Eq. (8), we can obtain an equation

$$\begin{bmatrix} \varphi_1^+ \\ \varphi_1^- \\ \varphi_2^+ \\ \varphi_2^- \\ \psi^+ \\ \psi^- \end{bmatrix} = \begin{bmatrix} g_{11} & m'_{12} & g_{12} & g_{13} & m'_{15} & g_{14} \\ g_{21} & m'_{22} & g_{22} & g_{23} & m'_{25} & g_{24} \\ g_{31} & m'_{32} & g_{32} & g_{33} & m'_{35} & g_{34} \\ g_{41} & m'_{42} & g_{42} & g_{43} & m'_{45} & g_{44} \\ g_{51} & m'_{52} & g_{52} & g_{53} & m'_{55} & g_{54} \\ g_{61} & m'_{62} & g_{62} & g_{63} & m'_{65} & g_{64} \end{bmatrix} \begin{bmatrix} u_B(z_0)/k \\ -P_f(z_j)/\omega^2 \\ \tau_P(z_0)/\omega^2 \\ u_P(z_0)/k \\ 0 \\ \tau_B(z_0)/\omega^2 \end{bmatrix}. \tag{11}$$

Based on boundary conditions on the free surface that the stress equals zero and the radial conditions on the depth direction that there is no ascending waves, the dispersion function of Rayleigh waves can be obtained in this case as

$$\begin{vmatrix} g_{21} & m'_{22} & g_{23} \\ g_{41} & m'_{42} & g_{43} \\ g_{61} & m'_{62} & g_{63} \end{vmatrix} = 0 \quad . \tag{12}$$

2.2 Porous layer is on the top

Based on $\boldsymbol{\varphi} = M^{-1}\mathbf{S}$ and the transfer relation of the matrices, combined with the first four equations of Eq. (7), we can get

$$\begin{aligned} \boldsymbol{\varphi}(z_{j+1}) &= M_{j+1}^{-1} P(z_j, z_{j-1}) \cdots \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix} \cdot S_d(z_0) \\ &= G \cdot S_d(z_0) \end{aligned}, \tag{13}$$

The G here is a matrix of 6×4.

With the propagation condition that there is only descending wave in the bottom layer, and the fifth equation of Eq. (7), we can getthe dispersion function as

$$\begin{vmatrix} g_{21} & g_{24} & g_{25} \\ g_{41} & g_{44} & g_{45} \\ h_{51} & h_{54} & h_{55} \end{vmatrix} = 0 , \tag{14}$$

2.3 Porous layer is at any interlayer

Now the porous layer is at the jth layer, the upper and lower layers are all elastic, the relation of both side interface of porous layer can be written as

$$\mathbf{S}_d(z_j) = M_d^{-1}\lambda_d M_d \mathbf{S}_d(z_{j-1}) = H_d \mathbf{S}_d(z_{j-1}), \tag{15}$$

based on Eq. (15) and Eq. (7), we can get

$$S_d(z_j) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix} S_d(z_{j-1}) . \tag{16}$$

Because W_z equals zero both at the upper and lower interface, so

$$0 = h_{51}(u_B/k) + h_{52}(-P_f/\omega^2) + h_{53}(\tau_P/\omega^2) + h_{54}(u_P/k) + h_{56}(\tau_B/\omega^2)$$

that is

$$(-P_f/\omega^2)_{z_{j-1}} = -\frac{1}{h_{55}}(h_{51}, h_{53}, h_{54}, h_{56}) S(z_{j-1}), \tag{17}$$

substituteEq. (16) into Eq. (15),

$$S(z_j) = P_d S(z_{j-1}), \tag{18}$$

P_d is the equivalent transfer matrix of porous media, a 4 orders square matrix it is.

By the transfer matrix, we can obtain

$$\boldsymbol{\varphi}(z_{j+1}) = M_{j+1}^{-1} P_d(z_j, z_{j-1}) \cdots P(z_1, z_0) \cdot \mathbf{S}(z_0) = G \cdot \mathbf{S}(z_0), \tag{19}$$

based on the existence conditions of Rayleigh waves, we can obtain the dispersion function as follow

$$g_{21}g_{43} - g_{23}g_{41} = 0 . \tag{20}$$

2.4 Porous layers are at the top and bottom layers

In this case, the j+1th layer is considered as a porous half-space, and the jth layer is elastic, and all the others are assumed as porous layers.Therefore, in the bottom layer (i.e. j+1), there is

$$\boldsymbol{\varphi}_d(z_{j+1}) = M_d^{-1} \mathbf{S}_d(z_j) . \tag{21}$$

From Eq. (7) we can get that u_B, u_P, τ_B, τ_P can be written as

$$S(z_j) = P(z_j, z_{j-1}) \cdots \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix} S_d(z_0) = P'(z_j, z_0) \cdot \mathbf{S}_d(z_0) , \tag{22}$$

though $w_P(z_j) = 0$, $P_f(z_j)$ is unknown, then we can build a D matrix as the format of Eq. (10). Then as the same procession from Eq. (10) to Eq. (12), the dispersion function is obtained in the same format with Eq. (12).

We deduce the dispersion functions of Rayleigh waves in a layered half-space with porous layers locate at different positions. Almost all kinds of half-space with porous layers can be induced as the formats above. For indication that the matrix H in Eq. (5) is still used in processing the interface between porous layers.

3. OPTIMIZING THE TRANSFER FUNCTIONS

There is a losing significant digit issue during the calculation of transfer matrix algorithm. In elastic layered half-space, Abo-zena [1] and Menke [2] improved the format of transfer functions, made a new matrix Y, the Y matrix's transfer relation in layered half-space is

$$Y^{(1)} = P^T(z_N, z_0) Y^{N+1} P(z_N, z_0) . \tag{23}$$

It is an antisymmetric 4 orders square matrix, then the dispersion function can be written as

$$Y_{12}^{(1)} = 0 . \tag{24}$$

Zhang [3] further improved the Y matrix, and made the transfer matrix as follow:

$$E^{j-1} = F^j E^j , \tag{25}$$

where $F = U\lambda^* V$, the format of U, V and λ^* can be seen in Ref. [3]. By this way, the square terms of e is eliminated, the computable frequency range is greatly expanded. The dispersion function here is

$$E_6^{(1)} = 0 . \tag{26}$$

However, when porous layers exist in half-space, this algorithm is not suitable because of the different size of transfer matrix. If we use the original format of transfer matrix, it would be a big limitation of computable frequency range. Furthermore, the multiplication of transfer matrix of porous media causes the cubic terms of e, it is obvious that the computable frequency range would be less than that of elastic media. Focus on solving this problem, we do our research on it.

α, β, γ represent the 2nd, 4th, and 6th row of $(M_d)_{N+1}^{-1}$, respectively. Then, define a set of matrices as follows:

$$\begin{cases} Y_a^{N+1} = \alpha \\ Y_a^N = Y_a^{N+1} H_d(z_N, z_{N-1}) \\ \vdots \\ Y_a^1 = Y_a^2 \cdot H_d(z_1, z_0) \end{cases} \quad \begin{cases} Y_b^{N+1} = \beta^T \gamma - \gamma^T \beta \\ Y_b^N = H_d^T(z_N, z_{N-1}) Y_b^{N+1} H_d(z_N, z_{N-1}) \\ \vdots \\ Y_b^1 = H_d^T(z_1, z_0) Y_b^2 H_d(z_1, z_0) \end{cases} . \tag{26}$$

So the dispersion function of porous layered half-space is

$$Y_a^1(1) \cdot Y_b^1(4,5) - Y_a^1(4) \cdot Y_b^1(1,5) + Y_a^1(5) \cdot Y_b^1(1,4) = 0 , \tag{27}$$

Y_b^j are a set of antisymmetric matrices, hence, we can obtain a column matrix which is made of the upper triangular elements of Y_b^{N+1}, which is the *Y* matrix of the bottom layer. By this way, we build an *E* matrix as

$$E^{N+1} = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}]^T$$

It is a column matrix of 15 rows. In each layer of layered half-space, E^j is satisfied of the deductive relation as follow:

$$E^{j-1} = F^j \cdot E^j . \tag{28}$$

Then, by the transfer relation of Y_b^j and E, the transfer matrix F can be inferred, which is a 15 orders square matrix. Furthermore, the F matrix can be deduced to a multiplication of 3 square matrices as

$$F = U_d \lambda_d^* V_d . \tag{29}$$

By this way, the cubic terms of e are eliminated. It greatly improves the computable frequency range.

We use Newton-Raphson method to solve the dispersion functions of Rayleigh waves. The method needs a relative accurate initial value for each modes of Rayleigh waves. Therefore, we introduce the bisection method into the complicated plane, in order to get the complicated roots of the dispersion functions.

The figures of dispersion curves calculated by original and optimized transfer matrix are shown in Fig. 2. From these figures, we can see that the computable frequency range is only about 12 kHz for the original method (left), but it is up to 22 kHz for the optimized one (right). In addition, the original method would lose some modes because of the calculation mistake, the 4th and 5th modes are miss in the left figure of Fig. 2. The problem is well solved in the optimized method.

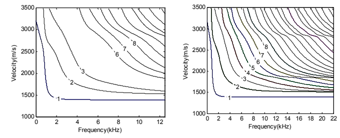

Figure2. The comparison of the dispersion curves calculated by the original and optimized transfer matrix.

4. CONCLUSION

The dispersion functions of Rayleigh waves in layered half-space consisted of elastic and porous layers. The dispersion functions are deducted by the boundary and propagation conditions of Rayleigh waves in cases of the porous layers located at different positions of half-space. The theoretical system of Rayleigh waves' excitation and propagation is improved in layered half-space with porous layers.

Then the transfer matrix of Rayleigh waves is optimized to solve the losing significant digit issue. By this means, the cubic terms of e are eliminated, the computable frequency range is greatly improved.

Finally, we introduce the bisection method into complicated plane to get the complicated roots of the dispersion functions. Using these roots as the initial values of Newton-Raphson method. The results are good.

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