APPLIED GEOPHYSICS, Vol.13, No.2 (June 2016), P. 333-342, 5 Figures.
DOI:10.1007/s11770-016-0536-2

Dispersion function of Rayleigh waves in porous layered half-space system*

Yan Shou-Guo♦1, Xie Fu-Li[2], Li Chang-Zheng[3], and Zhang Bi-Xing[1]

Abstract: Rayleigh wave exploration is based on an elastic layered half-space model. If practical formations contain porous layers, these layers need to be simplified as an elastic medium. We studied the effects of this simplification on the results of Rayleigh wave exploration. Using a half-space model with coexisting porous and elastic layers, we derived the dispersion functions of Rayleigh waves in a porous layered half-space system with porous layers at different depths, and the problem of transferring variables to matrices of different orders is solved. To solve the significant digit overflow in the multiplication of transfer matrices, we propose a simple, effective method. Results suggest that dispersion curves differ in a low-frequency region when a porous layer is at the surface; otherwise, the difference is small.
Keywords: layered media, porous media, Rayleigh waves, matrix optimization

Introduction

Porous formations are common in oil and gas exploration and are typically studied using Rayleigh waves based on a layered half-space model of elastic solid media (Zhang et al., 2004; O'Neil and Matsuoka, 2005; Lu and Zhang, 2006; Luo et al., 2007, 2008; Zhou et al., 2014). To better simulate field conditions and improve accuracy, porous layers should be considered in excitation and propagation mechanisms of Rayleigh waves in layered half-space systems. Yaroslav and David (2003) studied the dispersion of Rayleigh waves in a porous layered half-space system, and Zhou and Xia (2007) and Zhao et al. (2012) studied the dispersion in fluid-saturated porous media. Zhang et al. (2015) discussed the propagation of Rayleigh waves on the free surface of porous media with different connected conditions of fluid. Tajuddin and Ahmed (1991) computed the dispersion of Rayleigh waves in a porous two-layer half-space model. Parra and Xu (1994) investigated the dispersive characteristics of several modes of Rayleigh waves in layered media, and Xia et al. (2004) analyzed the dispersion characteristics of Rayleigh waves in open and closed elastic half-space systems with a saturated porous cover. Chai et al. (2015) studied the dispersion of multiple modes of Rayleigh waves in several layered porous media using the thin-layer method. However, most of these studies were based on a porous layered half-space model and focused on the regular pattern of Rayleigh waves and their dependence on medium properties. None of them considered the

Manuscript received by the Editor September 25, 2015; Revised manuscript received February 4, 2016.
*This work is supported by National Sciences Foundation (No.11174321, 11174322, and 11574343).
1. Institute of Acoustics, Chinese Academy of Sciences, Beijing 100190, China.
2. GoerTek Inc., Beijing 100190, China.
3. Yellow River Institute of Hydraulic Research, Zhengzhou 450003, China.
♦Correspondence author: Yan Shou-Guo (Email: yanshouguo@mail.ioa.ac.cn)

coexistence of an elastic solid and a porous layer or investigated the characteristics of Rayleigh waves in a different model.

On the other hand, the transfer-matrix method is widely used to study acoustic wave propagation in layered media because it is fast, simple, and yields physically meaningful results. Nevertheless, the precision of this method deteriorates at high frequencies. Furthermore, the loss of precision is maximized because of the presence of slow longitudinal waves and attenuation. Abo-Zena (1979) and Menke (1979) proposed a method to avoid the loss of precision in elastic solid layered half-space systems. The method was subsequently optimized by Zhang et al. (1996, 1998). All abovementioned methods are based on the relation between dispersion and the transfer matrix of elastic solid media, and they cannot readily account for the dispersion of Rayleigh waves in porous layered half-space systems.

This study aims to describe the dispersion of Rayleigh waves in porous layered half-space media and investigate the excitation and propagation of Rayleigh waves in porous layers. Moreover, an optimization method is proposed to improve the precision in the transfer-matrix method at high frequencies.

Boundary conditions and dispersion

A layered half-space model consists of arbitrarily mixed elastic solid and porous layers. Figure 1 shows the model and boundaries.

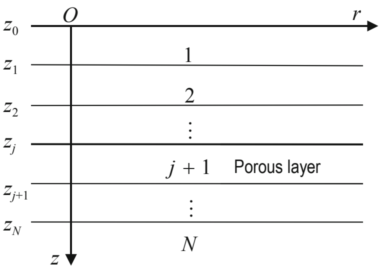


Fig.1 Layered half-space model with porous layers.

We adopt the BPC coordinate system for convenience (Ben-Menahem and Singh, 1968). **S** represents the displacement–stress vector in elastic media and is expressed as

$$\mathbf{S} = (u_B / k, \tau_P / \omega^2, u_P / k, \tau_B / \omega^2)^T, \tag{1}$$

where u_P and u_B are the displacement components and τ_P and τ_B are the stress components normal to the interface. Assuming that the jth layer is elastic, then the relation of the displacement–stress vector between the top and bottom interfaces is

$$\mathbf{S}_j(z_j) = \mathbf{M}_j \boldsymbol{\lambda}_j \mathbf{M}_j^{-1} \mathbf{S}_j(z_{j-1}), \tag{2}$$

where the derivations of $\mathbf{M}_j$, $\boldsymbol{\lambda}_j$, and $\mathbf{M}_j^{-1}$ are given in detail in Zhang et al. (1996).

In a porous medium, the displacement–stress vector is

$$\mathbf{S}^d = (u_B^d / k, -P_f^d / \omega^2, \tau_P^d / \omega^2, u_P^d / k, w_P^d / k, \tau_B^d / \omega^2)^T, \tag{3}$$

where P_f^d is the pore fluid pressure, u_P^d and u_B^d are the displacement components of the solid phase in the porous medium, w_P^d is the relative seepage displacement, and τ_P^d and τ_B^d are the stress components in the porous medium. Therefore, the recursive relation that can be obtained when the jth layer is porous is as follows:

$$\mathbf{S}_j^d(z_j) = \mathbf{M}_j^d \boldsymbol{\lambda}_j^d (\mathbf{M}_j^d)^{-1} \mathbf{S}_j^d(z_{j-1}). \tag{4}$$

Detailed derivations of $\mathbf{M}_j^d$, $\boldsymbol{\lambda}_j^d$, and $(\mathbf{M}_j^d)^{-1}$ are provided in Wu et al. (1993).

For convenience, equations (2) and (4) are transformed to

$$\begin{cases} \mathbf{S}_j(z_j) = \mathbf{P}_j \mathbf{S}_j(z_{j-1}) \\ \mathbf{S}_j^d(z_j) = \mathbf{H}_j \mathbf{S}_j^d(z_{j-1}) \end{cases}. \tag{5}$$

The transfer matrix $\mathbf{P}_j$ is a fourth-order square matrix for elastic layers, and $\mathbf{H}_j$ is a sixth-order square matrix for porous layers. Hence, we cannot directly compute the transfer function when porous and elastic layers coexist in a layered half-space system. Furthermore, boundary conditions for the free surface and acoustic propagation are different; thus, we need to derive new dispersion functions for the different relative positions of porous and elastic layers.

In the (b, p, c) coordinate system, there is no coupled relation between P–SV- and SH- waves; hence, they can be separately processed in the same manner. Consequently, we only discuss P–SV- waves. Vectors in elastic and porous media are

$$\begin{cases} \boldsymbol{\varphi}(z) \\ \quad = (Ae^{iaz}, Be^{-iaz}, Ce^{ibz}, De^{-ibz})^T = (\phi^+, \phi^-, \psi^+, \psi^-)^T \\ \boldsymbol{\varphi}^d(z) \\ \quad = (A_{m1}e^{a_1 z}, B_{m1}e^{-a_1 z}, A_{m2}e^{a_2 z}, B_{m2}e^{-a_2 z}, C_m e^{b_1 z}, D_m e^{-b_1 z})^T \\ \quad = (\phi_1^+, \phi_1^-, \phi_2^+, \phi_2^-, \psi_1^+, \psi_1^-)^T \end{cases}, \tag{6}$$

where ϕ and ψ represent the potential functions of P- and SV- waves in the elastic medium, respectively, ϕ_1 and ϕ_2 represent the first and second longitudinal wave potential functions, respectively, and ψ_1 represents the SV-wave potential function in the porous medium. Superscripts "+"and "–" denote wave propagation along and opposite to the z axis.

When a porous layer is the j + 1th layer，boundary conditions for z_j are

$$\begin{cases} u_B^j(z_j) = u_B^{j+1}(z_j) \\ u_P^j(z_j) = u_P^{j+1}(z_j) \\ \tau_P^j(z_j) = \tau_P^{j+1}(z_j) \\ \tau_B^j(z_j) = \tau_B^{j+1}(z_j) \\ \quad 0 = w_P^{j+1}(z_j) \end{cases} . \tag{7}$$

When a porous layer is located at different depths, there are two types of boundary conditions (pressure release surface)

$$\begin{cases} \tau_P(z_0) = 0 \\ \tau_B(z_0) = 0 \end{cases}, \tag{8a}$$

$$\begin{cases} P_f^d(z_0) = 0 \\ \tau_P^d(z_0) = 0 \\ \tau_B^d(z_0) = 0 \end{cases} . \tag{8b}$$

Because there is no upward-moving wave, there are two propagation conditions for acoustic waves in a bottom layer

$$\begin{cases} \phi^- = 0 \\ \psi^- = 0 \end{cases}, \tag{9a}$$

$$\begin{cases} \phi_1^- = 0 \\ \phi_2^- = 0 \\ \psi_1^- = 0 \end{cases} . \tag{9b}$$

Using different combinations of equations (8) and (9), one can obtain different dispersion functions for Rayleigh waves, which we discuss below.

Porous layer at the bottom

In this case, the porous layer is the j + 1th layer and all upper layers are elastic

$$\boldsymbol{\varphi}_d(z_{j+1}) = \mathbf{M}_d^{-1}\mathbf{S}_d(z_j). \tag{10}$$

Based on boundary conditions (first four terms of equation (7)), we obtain

$$\begin{bmatrix} u_B^{j+1}(z_j)/k \\ \tau_P^{j+1}(z_j)/\omega^2 \\ u_P^{j+1}(z_j)/k \\ \tau_B^{j+1}(z_j)/\omega^2 \end{bmatrix}_d = \mathbf{P}(z_j, z_{j-1})\cdots\mathbf{P}(z_1, z_0)\mathbf{S}(z_0) = \mathbf{P}(z_j, z_0)\mathbf{S}(z_0). \tag{11}$$

We construct a 6 × 4 matrix $\mathbf{G} = \mathbf{D}\cdot\mathbf{P}(z_j, z_0)$, where

$$\mathbf{D} = \begin{bmatrix} m'_{11} & m'_{13} & m'_{14} & m'_{16} \\ m'_{21} & m'_{23} & m'_{24} & m'_{26} \\ m'_{31} & m'_{33} & m'_{34} & m'_{36} \\ m'_{41} & m'_{43} & m'_{44} & m'_{46} \\ m'_{51} & m'_{53} & m'_{54} & m'_{56} \\ m'_{61} & m'_{63} & m'_{64} & m'_{66} \end{bmatrix},$$

and $m'_{ij} = [\mathbf{M}_d^{-1}]_{ij}$. The definition of matrix **G** is varied in different case of this paper.

We substitute equation (11) and the fifth term of equation (7) into equation (10) and obtain

$$\begin{bmatrix} \phi_1^+ \\ \phi_1^- \\ \phi_2^+ \\ \phi_2^- \\ \psi_1^+ \\ \psi_1^- \end{bmatrix} = \begin{bmatrix} g_{11} & m'_{12} & g_{12} & g_{13} & m'_{15} & g_{14} \\ g_{21} & m'_{22} & g_{22} & g_{23} & m'_{25} & g_{24} \\ g_{31} & m'_{32} & g_{32} & g_{33} & m'_{35} & g_{34} \\ g_{41} & m'_{42} & g_{42} & g_{43} & m'_{45} & g_{44} \\ g_{51} & m'_{52} & g_{52} & g_{53} & m'_{55} & g_{54} \\ g_{61} & m'_{62} & g_{62} & g_{63} & m'_{65} & g_{64} \end{bmatrix} \cdot \begin{bmatrix} u_B(z_0)/k \\ -P_f(z_j)/\omega^2 \\ \tau_P(z_0)/\omega^2 \\ u_P(z_0)/k \\ 0 \\ \tau_B(z_0)/\omega^2 \end{bmatrix}, \tag{12}$$

where $g_{ij}\,(i = 1,...,6,\ j = 1,...,4)$ are elements of matrix **G**.

Based on boundary conditions on the free surface where the stress is zero and radial conditions in the depth direction with no ascending waves,

$$\begin{bmatrix} g_{21} & m'_{22} & g_{23} \\ g_{41} & m'_{42} & g_{43} \\ g_{61} & m'_{62} & g_{63} \end{bmatrix} \begin{bmatrix} u_B/k \\ -P_f/\omega^2 \\ u_P/k \end{bmatrix} = 0, \tag{13}$$

and the dispersion function of Rayleigh waves, in this case, is

$$\begin{vmatrix} g_{21} & m'_{22} & g_{23} \\ g_{41} & m'_{42} & g_{43} \\ g_{61} & m'_{62} & g_{63} \end{vmatrix} = 0. \tag{14}$$

Porous layer at the top

Based on $\boldsymbol{\varphi} = \mathbf{M}^{-1}\mathbf{S}$ and the transfer relation of matrices, we combined the first four equations of equation (7) and obtained

$$\boldsymbol{\varphi}(z_{j+1}) = \mathbf{M}_{j+1}^{-1}\mathbf{P}(z_j, z_1) \cdot \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix} \cdot \begin{bmatrix} u_B(z_0)/k \\ -P_f(z_0)/\omega^2 \\ \tau_P(z_0)/\omega^2 \\ u_P(z_0)/k \\ w_P(z_0)/k \\ \tau_B(z_0)/\omega^2 \end{bmatrix} = \mathbf{G} \cdot \begin{bmatrix} u_B(z_0)/k \\ -P_f(z_0)/\omega^2 \\ \tau_P(z_0)/\omega^2 \\ u_P(z_0)/k \\ w_P(z_0)/k \\ \tau_B(z_0)/\omega^2 \end{bmatrix}, \tag{15}$$

where **G** is a 4 × 6 matrix.

When there are only descending waves in the bottom layer, based on the boundary conditions of the free surface and the fifth term in equation (7), the dispersion function is

$$\begin{bmatrix} g_{21} & g_{24} & g_{25} \\ g_{41} & g_{44} & g_{45} \\ h_{51} & h_{54} & h_{55} \end{bmatrix} \begin{bmatrix} u_B/k \\ u_P/k \\ w_P/k \end{bmatrix} = 0, \tag{16}$$

the dispersion function is

$$\begin{vmatrix} g_{21} & g_{24} & g_{25} \\ g_{41} & g_{44} & g_{45} \\ h_{51} & h_{54} & h_{55} \end{vmatrix} = 0. \tag{17}$$

Porous layer at any depth

Wu et al. (1993) discussed this condition and assumed that the jth layer is porous, upper and lower layers are elastic, and the relation at both interfaces of a porous layer is given by the second term in equation (5). Then, combining with equation (7), we obtain

$$\begin{bmatrix} u_B/k \\ \tau_P/\omega^2 \\ u_P/k \\ \tau_B/\omega^2 \end{bmatrix}_{z_j} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix} \cdot \begin{bmatrix} u_B/k \\ -P_f/\omega^2 \\ \tau_P/\omega^2 \\ u_P/k \\ w_P/k \\ \tau_B/\omega^2 \end{bmatrix}_{z_{j-1}}. \tag{18}$$

Because w_z equals zero at upper and lower interfaces,

$$0 = h_{51}(u_B/k) + h_{52}(-P_f/\omega^2) + h_{53}(\tau_P/\omega^2) + h_{54}(u_P/k) + h_{56}(\tau_B/\omega^2),$$

that is,

$$(-P_f/\omega^2)_{z_{j-1}} = -\frac{1}{h_{52}}(h_{51}, h_{53}, h_{54}, h_{56}) \begin{bmatrix} u_B/k \\ \tau_P/\omega^2 \\ u_P/k \\ \tau_B/\omega^2 \end{bmatrix}_{z_{j-1}}. \tag{19}$$

By substituting equation (19) into equation (18), we obtain

$$\begin{bmatrix} u_B/k \\ \tau_P/\omega^2 \\ u_P/k \\ \tau_B/\omega^2 \end{bmatrix}_{z_j} = \mathbf{P}_d \begin{bmatrix} u_B/k \\ \tau_P/\omega^2 \\ u_P/k \\ \tau_B/\omega^2 \end{bmatrix}_{z_{j-1}}, \tag{20}$$

where

$$\mathbf{P}_d = \begin{bmatrix} h_{11} - \frac{h_{12}h_{51}}{h_{55}} & h_{13} - \frac{h_{12}h_{53}}{h_{55}} & h_{14} - \frac{h_{12}h_{54}}{h_{55}} & h_{16} - \frac{h_{12}h_{56}}{h_{55}} \\ h_{31} - \frac{h_{32}h_{51}}{h_{55}} & h_{33} - \frac{h_{32}h_{53}}{h_{55}} & h_{34} - \frac{h_{32}h_{54}}{h_{55}} & h_{36} - \frac{h_{32}h_{56}}{h_{55}} \\ h_{41} - \frac{h_{42}h_{51}}{h_{55}} & h_{43} - \frac{h_{42}h_{53}}{h_{55}} & h_{44} - \frac{h_{42}h_{54}}{h_{55}} & h_{46} - \frac{h_{42}h_{56}}{h_{55}} \\ h_{61} - \frac{h_{62}h_{51}}{h_{55}} & h_{63} - \frac{h_{62}h_{53}}{h_{55}} & h_{64} - \frac{h_{62}h_{54}}{h_{55}} & h_{66} - \frac{h_{62}h_{56}}{h_{55}} \end{bmatrix}.$$

$\mathbf{P}_d$ is the equivalent transfer matrix of porous media; it is a fourth-order square matrix.

Based on the transfer matrix, we can obtain

$$\boldsymbol{\varphi}(z_{j+1}) = \mathbf{M}_{j+1}^{-1}\mathbf{P}_d(z_j, z_{j-1})\cdots\mathbf{P}(z_1, z_0)\cdot\mathbf{S}(z_0) = \mathbf{G}\cdot\mathbf{S}(z_0), \tag{21}$$

and the dispersion function of Rayleigh waves is

$$\begin{vmatrix} g_{21} & g_{23} \\ g_{41} & g_{43} \end{vmatrix} = 0. \tag{22}$$

Porous layers at the top and bottom

In this case, the $j+1^{\text{th}}$ layer is porous, jth layer is elastic, and remaining layers are assumed to be porous. Therefore, in the bottom layer, i.e., $j+1$, the relation is that of equation (10). From equation (7), we express u_B, u_P, τ_B, and τ_P as

$$\begin{bmatrix} u_B(z_j)/k \\ \tau_P(z_j)/\omega^2 \\ u_P(z_j)/k \\ \tau_B(z_j)/\omega^2 \end{bmatrix} = \mathbf{P}(z_j, z_{j-1})\cdots \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix} \begin{bmatrix} u_B(z_0)/k \\ -P_f(z_0)/\omega^2 \\ \tau_P(z_0)/\omega^2 \\ u_P(z_0)/k \\ w_P(z_0)/k \\ \tau_B(z_0)/\omega^2 \end{bmatrix} = \mathbf{P}'(z_j, z_0)\cdot\mathbf{S}_d(z_0). \tag{23}$$

Although $w_P(z_j) = 0, P_f(z_j)$ is unknown, we construct the $\mathbf{D}$ matrix as follows:

$$\mathbf{D} = \begin{bmatrix} m'_{11} & m'_{13} & m'_{14} & m'_{16} \\ m'_{21} & m'_{23} & m'_{24} & m'_{26} \\ m'_{31} & m'_{33} & m'_{34} & m'_{36} \\ m'_{41} & m'_{43} & m'_{44} & m'_{46} \\ m'_{51} & m'_{53} & m'_{54} & m'_{56} \\ m'_{61} & m'_{63} & m'_{64} & m'_{66} \end{bmatrix},$$

where the sixth-order square matrix $\mathbf{G} = \mathbf{D}\cdot\mathbf{P}'(z_j, z_0)$ and $m'_{ij} = [\mathbf{M}_d^{-1}]_{ij}$. Then, the dispersion function is

$$\boldsymbol{\varphi}_d(z_{j+1}) = \begin{bmatrix} g_{11} & m'_{12} & g_{13} & g_{14} & m'_{15} & g_{16} \\ g_{21} & m'_{22} & g_{23} & g_{24} & m'_{25} & g_{26} \\ g_{31} & m'_{32} & g_{33} & g_{34} & m'_{35} & g_{36} \\ g_{41} & m'_{42} & g_{43} & g_{44} & m'_{45} & g_{46} \\ g_{51} & m'_{52} & g_{53} & g_{54} & m'_{55} & g_{56} \\ g_{61} & m'_{62} & g_{63} & g_{64} & m'_{65} & g_{66} \end{bmatrix} \begin{bmatrix} u_B(z_0)/k \\ -P_f(z_j)/\omega^2 \\ \tau_P(z_0)/\omega^2 \\ u_P(z_0)/k \\ 0 \\ \tau_B(z_0)/\omega^2 \end{bmatrix}. \tag{24}$$

Based on wave propagation conditions and boundary conditions of the free surface, we obtain

$$\begin{bmatrix} g_{21} & m'_{22} & g_{24} \\ g_{41} & m'_{42} & g_{44} \\ g_{61} & m'_{62} & g_{64} \end{bmatrix} \begin{bmatrix} u_B(z_0)/k \\ -P_f(z_j)/\omega^2 \\ u_P(z_0)/k \end{bmatrix} = 0. \tag{25}$$

Because $w_P(z_1) = 0$ at the bottom of the first porous layer, we obtain

$$\mathbf{H}_1(5,1)\cdot u_B(z_0) + \mathbf{H}_1(5,4)\cdot u_P(z_0) + \mathbf{H}_1(5,5)\cdot w_P(z_0) = 0. \tag{26}$$

where $w_p(z_0)$ of the surface is not an independent variable, and the dispersion function of Rayleigh waves is

$$\begin{vmatrix} g_{21} & m'_{22} & g_{24} \\ g_{41} & m'_{42} & g_{44} \\ g_{61} & m'_{62} & g_{64} \end{vmatrix} = 0. \tag{27}$$

We derive the dispersion functions of Rayleigh waves in a layered half-space system with porous layers at different depths. Many types of half-spaces with porous layers can be derived, as described above, and matrix $\mathbf{H}$ in equation (5) is used to process the interface between porous layers.

Optimizing transfer functions

There is a loss of significant digits in the calculations of the transfer matrix algorithm. For an elastic layered half-space system, Abo-Zena (1979) and Menke (1979) improved the format of the transfer functions, and constructed a new matrix $\mathbf{Y}$

$$\mathbf{Y}^{(1)} = \mathbf{P}^T(z_N, z_0)\mathbf{Y}^{N+1}\mathbf{P}(z_N, z_0). \tag{28}$$

$\mathbf{Y}$ is an antisymmetric fourth-order square matrix and consequently the dispersion function is

$$\mathbf{Y}_{12}^{(1)} = 0. \tag{29}$$

Zhang et al. (1996) further improved the **Y** matrix and constructed the following transfer matrix

$$\mathbf{E}^{j-1} = \mathbf{F}^j \mathbf{E}^j, \tag{30}$$

where $\mathbf{F} = \mathbf{U}\boldsymbol{\lambda}^*\mathbf{V}$, and **U**, **V**, and $\boldsymbol{\lambda}^*$ are given in Zhang et al. (1996). Thus, the square terms in **E** are eliminated and the computational frequency range is expanded. The dispersion function is

$$\mathbf{E}_6^{(1)} = 0. \tag{31}$$

However, when porous layers exist in a half-space system, this algorithm is not suitable because of the variable size of the transfer matrix. If we use the original transfer matrix, this will limit the frequency range in the computations. Furthermore, the multiplication of the transfer matrix of porous media creates cubic terms in **E** and the frequency range is clearly smaller than that of the elastic medium. .

To solve this problem, we optimize the transfer matrix based on the Zhang et al. (1996) method. Based on the transfer relation and dispersion function of a porous layered half-space system, in the N + 1st layer,

$$\boldsymbol{\varphi}_d(z_{N+1}) = \mathbf{M}_d^{-1}(z_{N+1})\mathbf{H}_d(z_N, z_0)\mathbf{S}_d(z_0). \tag{32}$$

We assume that α, β, and γ represent the 2nd, 4th, and 6th row of $(\mathbf{M}_d)_{N+1}^{-1}$, $\mathbf{I}_1 = [1,0,0,0,0,0]^T, \mathbf{I}_4 = [0,0,0,1,0,0]^T$, and $\mathbf{I}_5 = [0,0,0,0,1,0]^T$, respectively. Thus, the dispersion function of the porous layered half-space system is

$$\begin{vmatrix} \alpha\mathbf{H}_d\mathbf{I}_1 & \alpha\mathbf{H}_d\mathbf{I}_4 & \alpha\mathbf{H}_d\mathbf{I}_5 \\ \beta\mathbf{H}_d\mathbf{I}_1 & \beta\mathbf{H}_d\mathbf{I}_4 & \beta\mathbf{H}_d\mathbf{I}_5 \\ \gamma\mathbf{H}_d\mathbf{I}_1 & \gamma\mathbf{H}_d\mathbf{I}_4 & \gamma\mathbf{H}_d\mathbf{I}_5 \end{vmatrix} = 0. \tag{33}$$

From equation (33), we obtain

$$\begin{aligned} &\alpha\mathbf{H}_d\mathbf{I}_1[\mathbf{I}_4^T\mathbf{H}_d^{\ T}(\beta^T\gamma - \gamma^T\beta)\mathbf{H}_d\mathbf{I}_5] \\ &-\alpha\mathbf{H}_d\mathbf{I}_4[\mathbf{I}_1^T\mathbf{H}_d^{\ T}(\beta^T\gamma - \gamma^T\beta)\mathbf{H}_d\mathbf{I}_5] \\ &+\alpha\mathbf{H}_d\mathbf{I}_5[\mathbf{I}_1^T\mathbf{H}_d^{\ T}(\beta^T\gamma - \gamma^T\beta)\mathbf{H}_d\mathbf{I}_4] = 0. \end{aligned} \tag{34}$$

Then, we define the following set of matrices

$$\begin{cases} \mathbf{Y}_a^{N+1} = \alpha \\ \mathbf{Y}_a^N = \mathbf{Y}_a^{N+1}\mathbf{H}_d(z_N, z_{N-1}) \\ \vdots \\ \mathbf{Y}_a^1 = \mathbf{Y}_a^2 \cdot \mathbf{H}_d(z_1, z_0) \end{cases},$$

$$\begin{cases} \mathbf{Y}_b^{N+1} = \beta^T\gamma - \gamma^T\beta \\ \mathbf{Y}_b^N = \mathbf{H}_d^T(z_N, z_{N-1})\mathbf{Y}_b^{N+1}\mathbf{H}_d(z_N, z_{N-1}) \\ \vdots \\ \mathbf{Y}_b^1 = \mathbf{H}_d^T(z_1, z_0)\mathbf{Y}_b^2\mathbf{H}_d(z_1, z_0) \end{cases}.$$

Consequently, the dispersion function of the porous layered half-space system is

$$\mathbf{Y}_a^1(1)\cdot\mathbf{Y}_b^1(4,5) - \mathbf{Y}_a^1(4)\cdot\mathbf{Y}_b^1(1,5) + \mathbf{Y}_a^1(5)\cdot\mathbf{Y}_b^1(1,4) = 0, \tag{35}$$

where $\mathbf{Y}_b^{\,j}$ is a set of antisymmetric matrices and the expression for the bottom layer (N + 1st layer) is

$$\mathbf{Y}_b^{N+1} = \begin{bmatrix} 0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ -y_1 & 0 & y_6 & y_7 & y_8 & y_9 \\ -y_2 & -y_6 & 0 & y_{10} & y_{11} & y_{12} \\ -y_3 & -y_7 & -y_{10} & 0 & y_{13} & y_{14} \\ -y_4 & -y_8 & -y_{11} & -y_{13} & 0 & y_{15} \\ -y_5 & -y_9 & -y_{12} & -y_{14} & -y_{15} & 0 \end{bmatrix}.$$

Hence, we obtain a column matrix that consists of the upper triangular elements of $\mathbf{Y}_b^{N+1}$, which is the **Y** matrix of the bottom layer. Next, we build the **E** matrix as

$$\mathbf{E}^{N+1} = \left[y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}\right]^T.$$

This is a column matrix of 15 rows. In each layer of the layered half-space system, $\mathbf{E}^j$ satisfies the following relation

$$\mathbf{E}^{j-1} = \mathbf{F}^j \cdot \mathbf{E}^j. \tag{36}$$

Then, the fifteenth-order square matrix **F** is derived and expressed as

$$\mathbf{F}_d = \mathbf{U}_d\boldsymbol{\lambda}_d^*\mathbf{V}_d, \tag{37}$$

where $\mathbf{U}_d$, $\boldsymbol{\lambda}_d^*$, and $\mathbf{V}_d$ are fifteenth-order square matrices.

In this manner, the cubic terms of **E** are eliminated. This consequently increases the frequency range in the computations. However, the method is applicable to media with porous layers only; for, it is hard to derive a uniform transfer matrix for a layered half-space system with coexisting elastic and porous layers because the number of waves differs in elastic and porous media. In the case the porous layer is inside the half-space, matrices $\mathbf{E}^{j-1} = \mathbf{F}^j\mathbf{E}^j$ and $\mathbf{Y}^{j-1} = (\mathbf{P}_d^j)^T\mathbf{Y}^j\mathbf{P}_d^j$ can be combined, and the **F** matrix in the elastic layers and the **Y** matrix in the porous layers can be used, avoiding the loss of precision.

Generally, the bisection method is used in solving the dispersion function in an elastic half-space system. The method searches the roots of all modes of Rayleigh waves without a priori knowledge; however, it only uses real functions and the computations are slow. Biot porous media in which acoustic waves attenuate, the dispersion function of the porous media is complex and the roots too. The Newton–Raphson method is more suitable to solve complex dispersion functions if we input accurate initial values. The bisection method can be used to search for the initial values in a particular frequency but, because the traditional bisection method cannot use complex functions, we expand the bisection method to the complex wavenumber plane and combine it with the Newton–Raphson method to solve the dispersion function of Rayleigh waves in a layered half-space system with porous layers. Parra and Xu (1994) discussed how to solve the dispersion function in the complex plane. First, the complex wavenumber plane is divided into grids and we then use the SIMPLEX method (Nelder and Mead, 1965) to solve the dispersion function. This is somewhat complex; thus, we use a simpler and more effective method to search for the solution.

1. Find the rough range of the roots, and divide the complex wavenumber plane into a grid.
2. The real and imaginary terms of the dispersion function of knots a, b, c, and d are calculated. If the value signs change between the knots, the solution is in the grid; otherwise, there is none.
3. If both signs of the real and imaginary terms of the dispersion function change, this means that the dispersion function may have a solution. Hence, we divide the grid into four subgrids and if the sign in each subgrid changes, the solution is constrained within a subgrid (Figure 2).

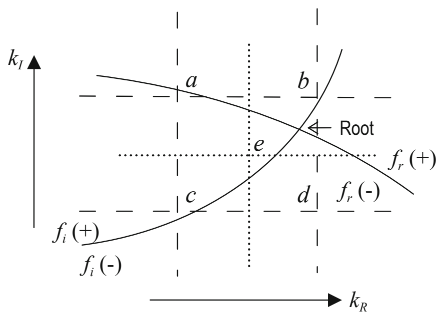


Fig.2 Bisection method in the complex plane.

4. Repeat step 3 until the size of the grid satisfies the precision, and the center of the grid is the solution if the signs of the real and imagine terms of the dispersion function change.
5. Take the solutions as the initial values for the Newton–Raphson method and continue the search.

Numerical simulation

Table 1 lists the medium parameters used in the computations. The parameters are for both the porous and equivalent elastic medium, with the porous-medium tortuosity taken as 3. The equivalent compressional- and shear-wave velocities ($\overline{V_P}$, $\overline{V_S}$) basically reflect the real velocity of the acoustic wave propagation in porous media. The average density ($\overline{\rho}$) represents the integrated density of the porous media. Therefore, it is reasonable to use elastic media with these parameters instead of the original porous media in the computations. In the numerical simulation, the porous layer can be located at any position in the half-space. By comparing the Rayleigh waves in the elastic half-space system, we evaluate the effect of the porous layer to the surface Rayleigh waves.

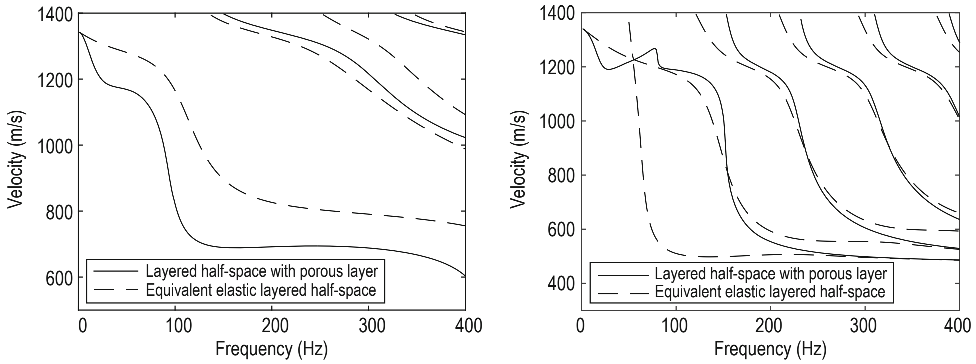
Table 1 Model parameters

Porous medium		T_1	T_2	B
Parameters	Units			
K_s	GPa	8.25	36	52.4
K_f	GPa	2.25	2.25	2.25
ρ_s	kg/m^3	2250	2250	2800
ρ_f	kg/m^3	1000	1000	1000
V_{bp}	m/s	1000	1700	2600
V_{bs}	m/s	500	950	1500
ϕ	-	0.2	0.2	0.2
k	μm^2	4.0	1.0	0.1
η_k	Pa·s	0.001	0.001	0.001

Equivalent elastic medium		T_1	T_2	B
Parameters	Units			
$\overline{V_P}$	m/s	1752	2540	3010
$\overline{V_S}$	m/s	474	900	1437
$\overline{\rho}$	kg/m^3	2000	2000	2440

There are two models that are typically used in numerical simulations and in practice. The first is the velocity-increasing model and the other is the low-velocity interlayer model, both of them with three layers. In the velocity-increasing model, the parameters of each layer from top to bottom are corresponding to parameters T1, T2, and B in Table 1. The parameters of the low-velocity interlayer model correspond from top to bottom to parameters T2, T1, and B, respectively.

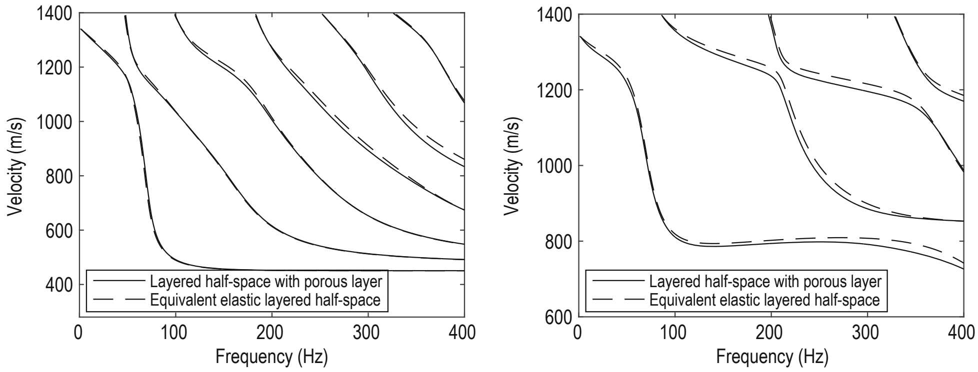
Figure 3 shows the case of a top porous layer with thickness of 0.4 m, a second layer with thickness of 3 m, and an infinite bottom layer. Figures 3a and 3b show the dispersion curves of the velocity-increasing and low-velocity interlayer model, respectively. The solid line is for a fluid-saturated top layer, and the dashed line is for the top layer using the equivalent elastic medium parameters instead of the porous medium parameters.



(a) Velocity-increasing model (b) Low-velocity interlayer model

Fig.3 Dispersion curves of Rayleigh waves for different models with a porous layer at the surface.

In Figure 3, for a porous layer is on top, there are significant differences between the solid and dashed lines. In Figure 3a, the surface porous layer causes the velocity of the first mode of Rayleigh waves to decrease significantly. Moreover, the velocity and cutoff frequency of higher order modes both increase. In the case of fluid-saturated porous layer, few modes are excited. In Figure 3b, when the top layer is porous, the dispersion of the first mode is abnormal, presumably, because the first- and second-order modes are combined. In higher order modes, all cutoff frequencies increase and the high-frequency velocity limit decreases.



(a) Velocity-increasing model (b) Low-velocity interlayer model

Fig.4 Dispersion curves of Rayleigh waves for different models with a porous interlayer.

Figure 4 shows the case of a porous layer in the middle of the three-layer half-space system with a 3 m thick top layer and 1 m thick porous middle layer. Figures 4a and 4b show the velocity-increasing and low-velocity interlayer model. In Figure 4a, the solid lines and dashed lines nearly coincide, and represent the middle porous and the equivalent elastic layer, respectively. In Figure 4b, there are small but not clear differences between the solid and dashed lines.

Figure 5 shows the dispersion curves for the model with a porous bottom layer, and 3 m thick top and 1 m thick middle layer. The dispersion curves are little affected by the porous layer.

Based on the comparisons, we can infer the following.

First, the dispersion curves of the model with the porous layer and with the equivalent elastic layer nearly coincide when the featured layer is at the bottom. There is little difference between the two kinds of models when the featured layer is in the middle. The difference is huge when the featured layer is at top. These differences may be caused by the energy distribution of the Rayleigh waves because we know that the energy of the Rayleigh waves always focuses on the surface; thus, the featured layer may strongly affect the dispersion when on top. In the case of the low-velocity interlayer, there may be a trap that affects the Rayleigh waves. Finally, when the featured layer is at the bottom, it hardly affects the Rayleigh waves.

In practice, the ground surface may resemble a porous medium. Thus, the effect of porous media needs to be considered when using Rayleigh waves in exploration. If only the velocity distribution is considered, it may be possible to describe a multilayered half-space system by using the elastic medium model. However, parameters such as porosity and permeability cannot be inverted. Therefore, it is important to study the propagation of Rayleigh waves in porous media and the interaction between the porous media and Rayleigh waves.

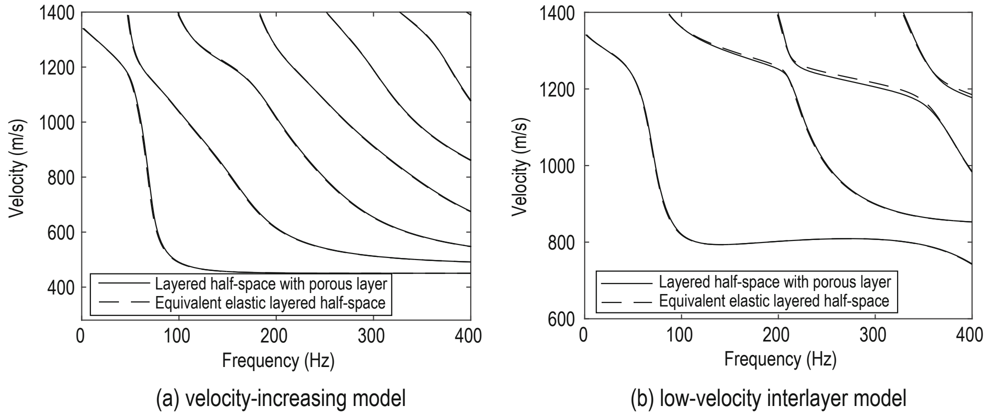


Fig.5 Dispersion curves of Rayleigh waves for different models with a porous layer at the bottom.

Conclusions

First, the dispersion function of Rayleigh waves is derived in a layered half-space system with a porous layer at different depths by the transfer-matrix method. The problem of nondirectly transferred variables owing to the different orders of the matrices is solved, and the computational model of Rayleigh waves in porous media is optimized.

Second, to solve the significant digit overflow in high frequencies, we derive a new fifteenth-order square matrix to eliminate the cubic terms. Moreover, we proposed a bisection algorithm in the complex plane to improve the computational efficiency.

Third, the comparison of the dispersion curves of Rayleigh waves for models with a porous and an equivalent elastic layer suggest that the dispersion is strongly affected when the porous layer is at the top and less when it is in the middle. When the porous layer is at the bottom, there is hardly any effect. The results also suggest that our method will improve exploration results that use Rayleigh waves.

Acknowledgements

This work is supported by the National Sciences Fo undation of China (No.11174321, 11174322 and 11574 343). We also appreciate Zhang Chengguang, Cheng Jianyuan, Cui Zhiwen and Lu Laiyu for the precious advis es.

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Yan Shou-Guo, Institute of Acoustics, Chinese Academy of Sciences, received his Ph. D. in 2014 from the University of Chinese Academy of Sciences. His main research fields are Nondestructive Testing and Evaluation, and acoustical propagation in complex media. Email: yanshouguo@mail.ioa.ac.cn